Sequences and Series

1. The first three of four integers are in an A.P. and the last three are in G.P.. Find these four numbers, given that the sum of the first and the last integers is 37 and the sum of the two integers in the middle is 36.

2. The 3rd, 6th and 12th terms of an A.P, are successive terms of a G.P. Show that 4th, 8th and 16th terms of the A.P. are also successive terms of a G.P.

3. Show that the sum of the odd numbers from 1 to (2n – 1) inclusive is n^2. Show that the sum of positive odd numbers smaller than 1002 that cannot be divided by 3 is 6×167^2.

4. The sum, S_n of the first n terms of the sequence $u_1, u_2, u_3, \ldots$ is $S_n = n(3n - a)$, where a is a constant.

(a) Find u_n in terms of a and n.

(b) Find the recurrence relation of u_n in the form of $u_{n+1} = f(u_n)$.

5. A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 4u_n + 7$ for $n \geq 1$. Write down the first four terms of the sequence,

Show that an explicit formula for u_r is given by $u_r = 1 + \frac{10}{3}[4^{r-1} - 1]$

6. Given $u_n = e^n - 1$. Prove that part of the sequence is a geometric progression. Hence find the value of $\sum_{r=1}^{n} u_r$

7. **(a)** Show that for a fixed number $x \neq 1$, $3x^2 + 3x^3 + \cdots + 3x^n$ is a geometric series and find its sum in terms of x and n.

(b) The series $T_n(x) = x + 4x^2 + 7x^3 + \cdots + (3n - 2)x^n$, for $x \neq 1$.
By considering $T_n(x) - xT_n(x)$ and using the result from (a), show that

$$T_n(x) = \frac{x + 2x^2 - (3n+1)x^{n+1} + (3n-2)x^{n+2}}{(1-x)^2}.$$

Hence, find the value of $\sum_{r=1}^{20} 2^r (3r - 2)$ and deduce the value of $\sum_{r=0}^{19} 2^{r+2} (3r + 1)$

8. **(a)** Use partial fractions to show that:

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\frac{5}{7\times9\times11}\ldots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{n(5n+7)}{6(2n+1)(2n+3)}$$

(b) State whether the series $\sum_{r=1}^{n}\frac{r+1}{(2r-1)(2r+1)(2r+3)}$ converges as $n\to\infty$ and if it does, find its sum to infinity.

9. Express $\frac{1}{(3r-2)(3r+1)}$ in partial fractions.

Show that $\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)}=\frac{1}{3}\left[1-\frac{1}{3n+1}\right]$.

Hence , find $\sum_{r=1}^{\infty}\frac{1}{(3r-2)(3r+1)}$.

10. Express $u_r=\frac{2}{(r+1)(r+3)}$ in partial fractions.

Using the result obtained,

(i) show that $u_r{}^2=-\frac{1}{r+1}+\frac{1}{r+3}+\frac{1}{(r+1)^2}+\frac{1}{(r+3)^2}$,

(ii) show that $\sum_{r=1}^{n}u_r=\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}$, and determine the values of

$\sum_{r=1}^{\infty}u_r$ and $\sum_{r=1}^{\infty}\left(u_{r+1}+\frac{1}{2^r}\right)$.